Chaotic control and synchronization for system identification

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Abstract

Research into applications of synchronized chaotic systems assumes that it will be necessary to build many different drive-response pairs, but little is known in general about designing chaotic systems, and even less is know about designing chaotic systems with specific properties. In this paper, I show that it is possible to create multiple drive-response pairs from one chaotic system by applying chaos control techniques to the drive and response systems. I show both numerical simulations and experimental work with chaotic circuits. I also test the response systems for ability to overcome noise or other interference.

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I. INTRODUCTION

There has been much work on using synchronized chaos [1–13] for applications such as communications or radar , but a constant assumption in all of this work has been that designing many different chaotic systems (for different transmitters, for example) would not be difficult. Simply designing new chaotic systems has not proven too difficult, as Sprott has shown [14], but designing chaotic systems with specific properties in mind is considerably more difficult. If one wants a chaotic system with a specific spectrum, for example, or a system that is robust to noise, or has particular Lyapunov exponents, there is no general theory for designing such systems. If the system must be built as an analog circuit, the problems are even harder. If designing one chaotic system is difficult, then designing many chaotic systems is even harder. In this paper, I show that one may use control techniques, such as the OGY technique [15], to create multiple drive-response pairs from one chaotic system. If the drive system is controlled to follow one sequence, and the response is controlled to follow the same sequence, synchronization will occur. If the drive and response are controlled to follow different sequences, synchronization will not occur. The resulting controlled systems will be periodic, rather than truly chaotic, but many desirable features of the chaotic system such as broad spectrum may be maintained. I demonstrate this control method in a numerical simulation and in a circuit experiment with a noise robust chaotic circuit.

II. CHAOTIC SYNCHRONIZATION

I assume a chaotic drive system of the form

$$\dot{\vec{x}} = f\left(\vec{x}\right) \tag{1}$$

and a response system of the form

$$\dot{\vec{y}} = g\left(\vec{y}\right) + h\left(\vec{x}\right) \tag{2}$$

where $\vec{x}$ and $\vec{y}$ are vectors, and $h(\vec{x})$ is a function of $\vec{x}$. The coupling in eq. (2) is a linear coupling, as are all types of coupling used in this paper, but other types of coupling are also possible.

For synchronization to occur, all the Lyapunov exponents of the dynamical system of eq. (2) must be negative. If the function g is identical to the function f, then identical synchronization is possible, otherwise the synchronization is said to be generalized synchronization. Detecting identical synchronization is easy, as signals in the response system will be almost the same (within experimental limits) as signals in the drive system. Detecting generalized synchronization is more difficult, and in fact there are many different definitions for generalized synchronization [12]. For this paper, we choose a response system with only one basin of attraction, and for synchronization we require only that the response system have all negative Lyapunov exponents. In this case, one may use an auxiliary system to detect synchronization [16]: two copies of the response system are built, and their outputs are compared to each other. If the outputs of the two response systems match (within experimental error), then the response is said to be synchronized to the drive.

III. CONTROL

Ott, Grebogi and Yorke (OGY) [15] showed that only small perturbations were necessary to control a chaotic system if the control kept the system near solutions of its equations of motion, such as unstable periodic orbits. Hayes et al. [17] later showed that one could encode information by using OGY control to switch between different trajectories of a chaotic system. As Hayes pointed out, the availability of multiple trajectories was a consequence of the positive entropy of a chaotic system. Hayes felt that this positive entropy should make chaotic signals natural information carriers. Hayes and others [11, 18, 19] have shown that one may use different states of the chaotic system as symbols, and control may be used to determine which symbol sequences are transmitted.

In this work, chaos control techniques are used to generate multiple different sequences from a chaotic system. These sequences have a finite length, so they must be repeated, but the sequences may be chosen long enough that they still have broad band spectra. A chaotic response system is also controlled so that it will synchronize only to one sequence from a chaotic transmitter, and not to any others.

IV. NUMERICAL WORK

I first demonstrate control and synchronization in a numerical experiment. The drive system is similar to the piecewise-linear Rossler system [20]:

$$\frac{dx_1}{dt} = -\alpha\left(0.05x_1 + 0.5x_2 + x_3\right)$$
$$\frac{dx_2}{dt} = -\alpha\left(-x_1 - 0.3x_2\right) \tag{3}$$
$$\frac{dx_3}{dt} = -\alpha\left(-g\left(x_1\right) + x_3\right)$$
$$g\left(x\right) = \left\{ \begin{array}{ll} m_1 x + b_2 & x \leq -x_0 \\ m_0 x & -x_0 < x < x_0 \\ m_1 x + b_1 & x \geq x_0 \end{array} \right\}$$

where $\alpha = 10.0, m_0 = 0.1, m_2 = 15.0, x_0 = 3.0, b_1 = x_0(m_0 - m_1)$, and $b_2 = -b_1$. Figure 1(a) is a plot of x_2 vs. x_1 , while 1(b) is a plot of x_3 vs. x_2 . These equations were integrated with a 4th order Runge-Kutta integrator with a time step of 0.04 s. While most previous control work caused chaotic systems to follow a set of unstable periodic orbits, control can also be used to make the system follow a chaotic trajectory. I allowed the system of eq. (3) to evolve freely in time, and recorded the values of x_1 and x_3 when x_2 crossed 0 in the positive direction. I created 2 such control sequences, each 100 cycles long. To make the system of eq. (3) follow either one of these sequences, when x_2 crossed 0 I set x_1 and x_3 equal to their corresponding values from the control sequence. After the first control point, the chaotic system is close to a natural chaotic trajectory, so the change in the system caused by resetting x_1 and x_3 was very small. At the end of the 100 cycle control sequence, the control sequence is repeated, so the resulting system is actually periodic with a period of 100.

The response system is a duplicate of the drive system. The response system is described by

$$\frac{dy_1}{dt} = -\alpha\left(0.05y_1 + 0.5y_2 + y_3\right)$$
$$\frac{dy_2}{dt} = -\alpha\left(-y_1 - 0.3y_2 + c\left(y_2 - x_2\right)\right) \tag{4}$$
$$\frac{dy_3}{dt} = -\alpha\left(-g\left(y_1\right) + y_3\right)$$
$$\frac{dz}{dt} = \alpha\left(\left|x_2 - y_2\right| - z\right)$$

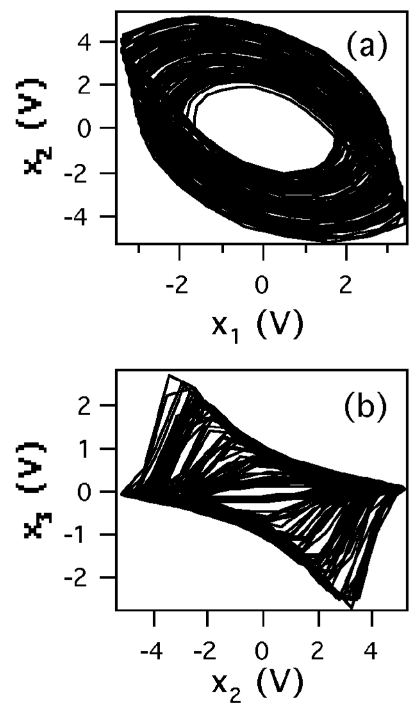

FIG. 1: Attractors for the chaotic system of eq. (3).

and the variable z is a measure of the average synchronization error and the coupling constant $c = 0.1$.

When the uncontrolled response system of eq. (4) is driven by the uncontrolled drive system of eq. (3), identical synchronization occurs; that is, y_1, y_2, and y_3 approach x_1, x_2, and x_3 . If drive and response are controlled by the same control sequence, identical synchronization will also occur, but the drive and response control sequences must be in phase with each other. The variable z in eq. (4) is used to help judge if the drive and response control sequences are in phase. With control on for the response system, when y_2 crosses 0 in the positive direction, the value of z is compared to some threshold. If z is less than the threshold, then it is assumed that drive and response are synchronized, and y_1 and y_3 are set to the appropriate values in the control sequence. If z exceeds the threshold, then it is assumed that the drive and response control sequences are out of phase, and so the phase of the response control sequence is advanced by 1 before control is applied. In a manner similar to digital code division multiple access (CDMA) [21], the response control sequence is advanced at a faster rate than the drive control sequence until synchronization is obtained.

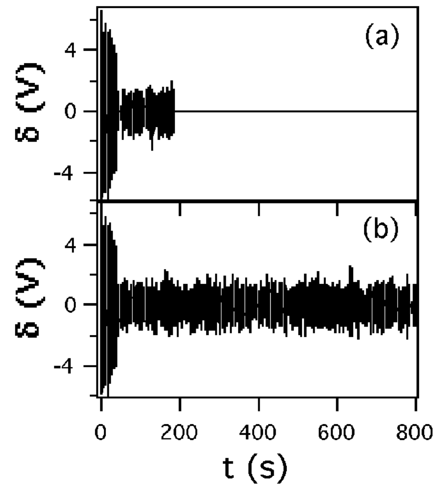

FIG. 2: Results of driving and controlling the response system of eq. (4). $\delta = x_1 - y_1$ is the difference between drive and response systems. In (a), the proper response control sequence was used, resulting in synchronization after a transient. In (b), the wrong response control sequence was used, resulting in no synchronization.

Figure 2 shows the results of the control when a threshold of $z = 0.2$ is used. Figure 2 is a plot of $\delta = x_1 - y_1$. In fig 2(a), the response system is free running ($c = 0$) for the first 40 s. At $t = 40$ s, c is set to 0.1 and control is applied to the response. Initially the drive and response control sequences are out of phase, so there is an initial transient (which should depend on the length of the control sequence) before synchronization is obtained at about 200 s. In fig. 2(b), different control sequences are used for drive and response. Once again, control is started at 40 s, but because drive and response control sequences are different, no synchronization is seen.

The utility of this signal recognition method will depend on how many different control sequences can be produced. It should be possible to estimate the number of different control sequences by assigning symbol sequences to the control sequences. First, a generating partition for the chaotic system must be found. While this is a difficult process in general, there are some recent methods for finding such partitions [22, 23]. The generating partition is then used to define a set of symbols; i.e., if the chaotic trajectory passes through one region, one particular symbol is produced, while if it passes through a different region, a different symbol may be produced. It may be that the chaotic system possess a grammar, so that not all symbol sequences occur. The number of different symbol sequences that occur for a given sequence length L should correspond to the number of possible control sequences.

This simple flow is useful to show how the control and synchronization method works, but it is not practical. Adding Gaussian white noise with an amplitude of 0.02 or greater destroys synchronization. Below I show a circuit example which is more robust to additive noise.

V. CIRCUIT EXPERIMENTS

The circuit used for these experiments is based on a chaotic system that maintains phase synchronization even when noise much larger than the transmitted signal is present [24, 25]. This system consists of a Rossler like chaotic circuit which operates in one frequency range coupled to a stable (nonoscillating) system which operates in a much lower frequency range. The separation of frequencies allows the lower frequency part of the response circuit to stay in phase synchronization to the lower frequency part of the drive system.

The circuits used were built using operational amplifiers, so they functioned as analog computers. The drive circuit may be approximately described by the equations

$$\begin{aligned}
\frac{dx_1}{dt} &= -\frac{1}{RC_1}\left(0.02x_1 + 0.5x_2 + 0.5\left|x_4\right|\right) \\
\frac{dx_2}{dt} &= -\frac{1}{RC_1}\left(-x_1 + 0.02x_2\right) \\
\frac{dx_3}{dt} &= -\frac{f\left(x_1\right)}{RC_2}\left(0.02x_3 + 0.5x_4 + x_5 + 0.1x_1\right) \\
\frac{dx_4}{dt} &= -\frac{f\left(x_1\right)}{RC_2}\left(-x_3 - 0.13x_4\right) \\
\frac{dx_5}{dt} &= -\frac{f\left(x_1\right)}{RC_2}\left(-g\left(x_3\right) + x_5\right) \\
g\left(x\right) &= \left\{\begin{array}{ll} 0 & x < 3 \\ 15\left(x-3\right) & x \geq 3 \end{array}\right\} \\
f\left(x\right) &= 1 + 0.2\left(x + 1.75\right)
\end{aligned} \tag{5}$$

where $R = 100\ \mathrm{k}\Omega$, $C_1 = 0.1\mu\mathrm{F}$, and $C_2 = 0.001\mu\mathrm{F}$. For these parameters, the signal x_1 has a frequency of approximately 10.5 Hz, while x_3 has a frequency of about 946 Hz. Figure 3(a) is a plot of x_2 vs. x_1, and 3(b) is a plot of x_4 vs. x_3. The function $f(x)$ serves to broaden the spectrum of the fast signals (x_3 through x_5).

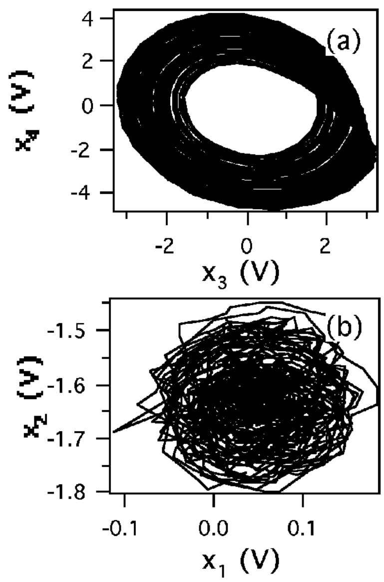

FIG. 3: Attractors for the chaotic circuit used to provide a driving signal. (a) is the attractor from the fast part of the circuit, while (b) is the attractor for the slow part of the circuit.

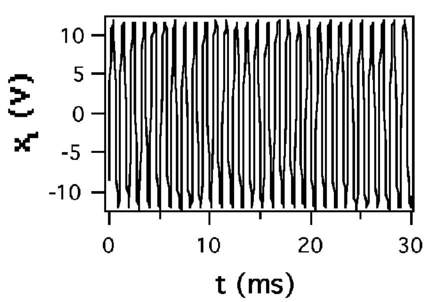

FIG. 4: Transmitted signal x_t produced by the chaotic driving circuit.

The signal that is actually transmitted is x_t defined by

$$\frac{dx_t}{dt} = -\frac{1}{RC_2}\left(sq\left(\frac{x_4}{x_3^2 + x_4^2}\right) + x_t\right) \tag{6}$$

where the $sq(x)$ function means that $sq(x)$ = 15 V if $x > 0$ and $sq(x)$ = -15 V if $x < 0$. The $sq(x)$ function was executed by an op amp with a very large gain. The integral was used as a low pass filter so that x_t was not a square wave.

Figure 4 is a plot of x_t as a function of time, while figure 5 is its power spectrum. The signal x_t has a constant envelope, which makes it more efficient to transmit, and makes it

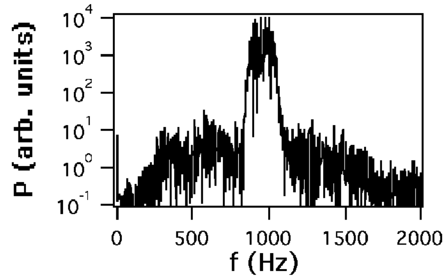

FIG. 5: Power spectrum of the signal x_t produced by the chaotic drive circuit.

easier to restore its amplitude to a known value after transmission.

The response circuit may be described by the equations

$$\begin{aligned}
\frac{dy_1}{dt} &= -\frac{1}{RC_1}\left(0.1y_1 + 0.5y_2 + 0.5\left|y_3\right|\right) \\
\frac{dy_2}{dt} &= -\frac{1}{RC_1}\left(-y_1 + 0.1y_2\right) \\
\frac{dy_3}{dt} &= -\frac{1}{RC_2}\left(0.02y_3 + 0.5y_4 + 0.1y_1\right) \\
\frac{dy_4}{dt} &= -\frac{1}{RC_2}\left(-y_3 - kx_t\right)
\end{aligned} \tag{7}$$

where R, C_1 and C_2 are the same as in eq. (5). The constant k is used to alter the amplitude of the transmitted signal x_t.

The response circuit does not match the drive circuit, which means that exact synchronization is not possible. In order to determine when generalized synchronization took place, the auxiliary system approach was used [26]. A second response circuit that was identical (within experimental error) was built. In order to improve the matching between circuits, resistors with a 1% tolerance were used, and a 20 turn potentiometer was used in the integrator for the y_1 signal to correct the time constant $1/RC_1$ for error in the capacitor value. The y_1 signals from the two response circuits were compared to determine if generalized synchronization was occurring.

Rather than try to control the drive circuit as in the numerical section, a 10,000 point signal x_t from the drive circuit was digitized at 20,000 points/s and played back through an arbitrary waveform generator. The playback rate was chosen so that the frequency of the signal from the arbitrary waveform generator matched the frequency of the original drive signal. Chaotic signals were recorded at 2 different times, resulting in 2 different chaotic sequences, labeled as *chaos1* and *chaos2* . The chaotic signals were played back with a peak to peak amplitude of 1.98 V, and the drive constant k in eq. (7) was set to 1.0.

For the control of the response circuit, the y_1 signal was first passed through a 1 μF capacitor to remove the DC component. This signal was then integrated by an op amp integrator to smooth out any residual ripple in y_1, producing the signal ψ :

$$\frac{d\psi}{dt} = -\frac{1}{RC_1}(y_1 + 0.1\psi) \tag{8}$$

where R and C_1 were previously defined. Several logic circuits were then used to give a short +5 V pulse when ψ crossed 0 in the negative direction.

In order to record the necessary control information, the response circuit was driven by *chaos1* or *chaos2*, and the value of y_1 was stored by a computer at the 0 crossings of ψ to create a response control sequence.

During control, the response driven by *chaos1* or *chaos2*. When ψ crossed 0 in the negative direction, the difference between y_1 and the corresponding signal from the matching auxiliary circuit, y_{1a}, was compared to a fixed threshold in the computer. If $|y_1 - y_{1a}| > 0.3$, it was assumed that the circuits were not synchronized, and the phase of the control sequence was advanced by 1. If the difference was less than the threshold, the control phase was not advanced. For either result, the computer then set y_1 for the circuit to the value in the control sequence, after which the control sequence phase was advanced. The sequences *chaos1* and *chaos2* corresponded to 5 cycles of the slow part of the circuit, so each control sequence had a length of 5.

Figure 6 is a plot of y_{1a} vs. y_1 when the arbitrary waveform generator is playing back *chaos1* and the response circuit is being controlled by the control sequence corresponding to *chaos1*. There are some occasional small departures from synchronization, but most of the time the 2 auxiliary systems are synchronized. Figure 7 is the same plot when the drive signal is *chaos2* but the control sequence still corresponds to *chaos1*. There is a definite loss of synchronization, so the pair of response circuits are able to recognize the difference between *chaos1* and *chaos2*.

This circuit can still recognize the difference between *chaos1* and *chaos2* when noise is present. The arbitrary waveform generator was used to produce a Gaussian white noise signal with a bandwidth of 50 kHz, which was added to the drive signal *chaos1*. Figure 8

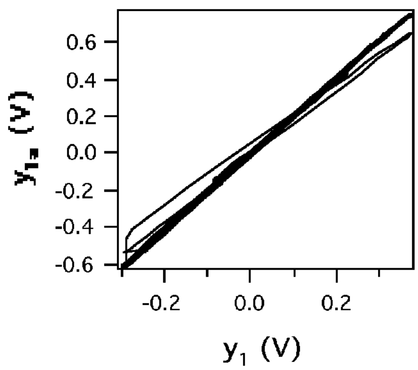

FIG. 6: Plot showing synchronization of the response circuit (y_1) and the auxiliary response circuit (y_{1a}), confirming generalized synchronization when the correct response control sequence for a particular drive signal is used.

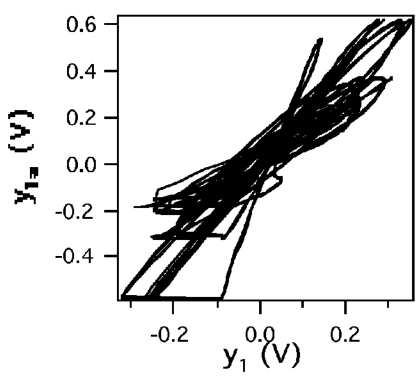

FIG. 7: Plot showing a lack of generalized synchronization between the response circuit (y_1) and the auxiliary response circuit (y_{1a}) when a response control sequence that does not correspond to the drive signal is used.

shows y_{1a} vs. y_1 when the drive signal is *chaos1* plus noise, with a signal power to noise power ratio of 0.7 (-1.4 dB), and the response system is controlled by the control sequence corresponding to *chaos1* . The synchronization is still recognizable when fig. 8 is compared to fig 7, where there was no noise, but the wrong control sequence was used. The cross correlation at 0 time lag between y_1 and y_{1a} when the wrong control sequence was used but no noise was present was 0.93, while the cross correlation when the correct control was used but the signal to noise ratio was 0.7 was 0.98.

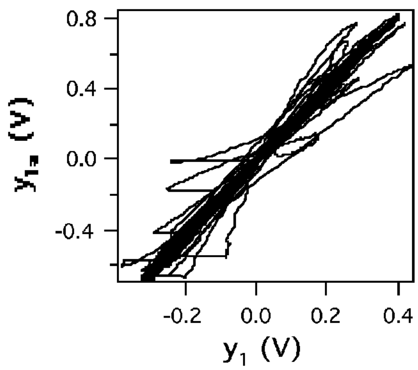

FIG. 8: Plot of the auxiliary response circuit (y_{1a}) vs. the response circuit (y_1) showing that generalized synchronization is maintained even when additive Gaussian white noise larger than the drive signal is present.

The effect of interference from another chaotic signal on the response circuits was also tested. A second arbitrary waveform generator was used to play back the *chaos2* signal, which was added to the *chaos1* signal. When both signals had the same amplitude, the cross correlation between y_1 and y_{1a} was 0.96. When the *chaos2* signal amplitude was 1.5 times the amplitude of the *chaos1* signal, the cross correlation dropped to 0.91, lower than the value when the wrong drive signal was used. The response circuits can reject some interference, but they have trouble if the interference is too similar to the driving signal.

VI. CONCLUSIONS

It is possible to control a self-synchronizing chaotic system so that it only synchronizes to the proper drive signal, and not a different signal derived from the same chaotic system. This observation was tested in numerical simulations and in circuit experiments. The control and synchronization procedure should make it easier to design multiple drive-response pairs, as it is not necessary to build a completely different chaotic circuit for each pair.

In the circuit experiments, the ability of a controlled response system to recognize a particular signal in the presence of noise or interference was tested. It has been shown in previous work that the noise robustness of similar 2 frequency circuits may be improved by increasing the separation between fast and slow frequencies [24]. Resistance to chaotic interference was not as good, but designing the chaotic drive system so that different output sequences were less similar to each other should increase the resistance to this type of interference. The control techniques used here also allow greater freedom in designing the transmitter, since the receiver no longer has to be a replica of the transmitter.

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